RICE STAT 533 / GSBS 1283, Spring 2020 Exam 1

DO NOT TURN OVER THIS TEST UNTIL YOU ARE TOLD TO DO SO

- You have 1 hour 15 minutes to complete this test.
- Show your work for each question. Partial credit will be given for partially worked out problems.

1. Suppose $X_1, \ldots, X_n \sim_{iid} f(x|\theta)$ from an exponential location model where

$$f(x|\theta) = e^{-(x-\theta)} \mathbb{1}_{x \geq \theta}.$$

We are interested in testing the hypothesis

$$\begin{aligned} H_0 &: \theta \leq \theta_0 \\ H_a &: \theta > \theta_0. \end{aligned}$$

(a) (10 points) Determine a size α likelihood ratio test for this hypothesis. The test should depend on the data through a 1-dimensional sufficient statistic. Write the test function T with an explicit cutoff based on this statistic (cutoff will depend on α).

Answer: The joint pdf is

$$f(\vec{x}|\theta) = \underbrace{e^{-\sum x_i}}_{h(x)} \underbrace{e^{n\theta} \mathbb{1}_{\theta \leq x_{(1)}}}_{g(T = X_{(1)}|\theta)}$$

Therefore by the factorization theorem $T = X_{(1)}$ is sufficient. The LR statistic is

$$\lambda(x) = \frac{\sup_{\theta \in \Theta_0} L(\theta|x)}{\sup L(\theta|x)}$$

The unrestricted MLE is $X_{(1)}$. If $X_{(1)} < \theta_0$, then the restricted MLE is also $X_{(1)}$. Otherwise the restricted MLE is θ_0. Thus we have

$$\lambda(x) = \begin{cases} 1 & : x_{(1)} \leq \theta_0 \\ e^{-n(x_{(1)} - \theta_0)} & : o.w. \end{cases}$$

and the LRT rejection region is

$$R = \{x : \lambda(x) \leq c\}$$

which is equivalent to

$$R = \{x : x_{(1)} > \underbrace{\theta_0 - \frac{\log c}{n}}_{\equiv d}\}$$

We need to compute d in the rejection region from part b). Using the density of the order statistic we can determine that the pdf for $X_{(1)}$ is

$$f_{X_{(1)}}(x|\theta) = n e^{-n(x-\theta)} \mathbb{1}_{x \geq \theta}$$

So

$$\alpha = \sup_{\theta \in \Theta_0} P_\theta(X \in R) = P_{\theta_0}(X_{(1)} > d) = \int_d^\infty f_{X_{(1)}}(x|\theta_0) dx$$

Computing the integral on the right we determine

$$\alpha = e^{-n(d - \theta_0)}$$

Thus

$$d = \theta_0 - \frac{\log \alpha}{n}.$$

(b) (3 points) Is this test UMP size α? Why or why not?

Answer: Yes. The model is not a member of the exponential family (support changes with θ), but it does have a MLR in $X_{(1)}$. Note that for $\theta_2 > \theta_1$, we have

$$\frac{f_{X_{(1)}}(x|\theta_2)}{f_{X_{(1)}}(x|\theta_1)} = \frac{ne^{-n(x-\theta_2)}\mathbf{1}_{x>\theta_2}}{ne^{-n(x-\theta_1)}\mathbf{1}_{x>\theta_1}} = e^{n(\theta_2-\theta_1)}\frac{\mathbf{1}_{x>\theta_2}}{\mathbf{1}_{x>\theta_1}}$$

which equals 0 for $x \leq \theta_2$ and a constant for $x > \theta_2$. Therefore $X_{(1)}$ has a MLR and rejecting for large $X_{(1)}$ represent UMP tests.

2. Suppose $X_i \sim_{iid} f(x|p)$ for $i = 1, \dots, n$ from a $Poisson(\theta)$ model where

$$f(x|\theta) = \frac{e^{-\theta}\theta^x}{x!}$$

We would like to test the hypotheses

$$\begin{aligned} H_0 &: \theta = \theta_0 \\ H_1 &: \theta \neq \theta_0 \end{aligned}$$

Suppose we put an exponential prior on θ conditional on the alternative, i.e.

$$\pi(\theta|\theta \in \Theta_1) = e^{-\theta}\mathbf{1}_{\theta>0}$$

(a) (10 points) Compute the Bayes factor for this model.

Answer: The Bayes factor and Bayes test depend only on a sufficient statistic. In this case $Y = \sum X_i \sim Poisson(n\theta)$ is sufficient. We have

$$\begin{aligned} m_1(y) &= \int_\theta f(y|\theta)\pi(\theta)d\theta \\ &= \int_\theta \frac{e^{-n\theta}(n\theta)^t}{y!}e^{-\theta} \\ &= \frac{n^y}{y!}\Gamma(y+1)\left(\frac{1}{n+1}\right)^{y+1} \underbrace{\int \frac{1}{\Gamma(y+1)(n+1)^{-(y+1)}}e^{-\theta(n+1)}\theta^y d\theta}_{=1} \\ &= \left(\frac{n}{n+1}\right)^y\left(\frac{1}{n+1}\right) \end{aligned}$$

where the integral equaling 1 is justified by the fact that this is a Gamma density, thus it integrates to 1. Therefore the Bayes factor is

$$\beta = \frac{m_0(y)}{m_1(y)} = \frac{\frac{e^{-n\theta_0}(n\theta_0)^y}{y!}}{\left(\frac{n}{n+1}\right)^y\left(\frac{1}{n+1}\right)}$$

(b) (3 points) Suppose the prior on the null model is π_0. Find the posterior probability of the null $\widehat{\pi}_0$.

Answer: Since

$$\frac{\widehat{\pi}_0}{1-\widehat{\pi}_0} = \beta\frac{\pi_0}{1-\pi_0}$$

we have

$$\widehat{\pi}_0 = \frac{\beta\frac{\pi_0}{1-\pi_0}}{1+\beta\frac{\pi_0}{1-\pi_0}}$$

3. (10 points) Let $X = (X_1, X_2, X_3) \sim Multi(n, p)$ from the multinomial distribution where $p_1 + p_2 + p_3 = 1$ and $p_j > 0$. Derive a Wald **or** Rao test statistic for the hypotheses

$$H_0\colon p_2 = 2p_3$$
$$H_1\colon p_2 \neq 2p_3$$

and state its limiting distribution.

Answer: Define the parameter to be $p = (p_2, p_3)$ and $p_1 \equiv 1 - p_2 - p_3$. Then the null can be represented as $H_0\colon R(p) = 0$ where $R(p) = (1, -2)p = p_2 - 2p_3$. Then we have $C = (1, -2)^T$. For the Wald, the unrestricted MLE is $\widehat{p} = (X_2/n, X_3/n)$ and the test statistic is

$$\begin{aligned} W_n &= R(\widehat{p})^T (C(\widehat{p})^T I_n^{-1}(\widehat{p}) C(\widehat{p}))^{-1} R(\widehat{p}) \\ &= \frac{(\widehat{p}_2 - 2\widehat{p}_3)^2}{(1, -2) I_n^{-1}(\widehat{p})(1, -2)^T} \end{aligned}$$

We have (see homework for more details of derivation)

$$I_n(\widehat{p}) = n\begin{pmatrix} \widehat{p_2}^{-1} & 0 \\ 0 & \widehat{p_3}^{-1} \end{pmatrix} - n\begin{pmatrix} \widehat{p_1}^{-1} & \widehat{p_1}^{-1} \\ \widehat{p_1}^{-1} & \widehat{p_1}^{-1} \end{pmatrix}$$

So

$$I_n^{-1}(\widehat{p}) = n^{-1}\begin{pmatrix} \widehat{p}_2 & 0 \\ 0 & \widehat{p}_3 \end{pmatrix} + n^{-1}\widehat{pp^T}$$

Plugging these into the Wald statistic we have

$$W_n = \frac{n(\widehat{p}_2 - 2\widehat{p}_3)^2}{(\widehat{p}_2 + 4\widehat{p}_3) + (\widehat{p}_2^2 - 2\widehat{p}_3)^2}$$

$W_n \to_d \chi_1^2$ by Theorem 6.6.

For the Rao test, restricted MLE is

$$\widetilde{p} = (2(X_2 + X_3)/3, X_2 + X_3/3)$$

The score function is

$$s_n(\widetilde{p}) = \left(\frac{n(X_2 - 2X_3)}{2(X_2 + X_3)}, \frac{n(2X_3 - X_2)}{X_2 + X_3} \right)$$

Noting that $R_n = s_n(\widetilde{p})^T I_n^{-1}(\widetilde{p}) s_n(\widetilde{p})$ and using the derivating of I_n^{-1} from the Wald we have

$$R_n = \frac{(X_2 - 2X_3)^2}{2(X_2 + X_3)} = n\frac{(\widetilde{p}_2 - 2\widetilde{p}_3)^2}{2(\widetilde{p}_2 + \widetilde{p}_3)}$$

$R_n \to_d \chi_1^2$ by Theorem 6.6.