Rice STAT 533/ GS01 1283 Homework 1 Solutions
January 23, 2020
Request: Please email the instructor if you find any mistakes in this document.

Section 6.6 Exercise 1: We do the case with $\alpha = 1$. The $\alpha = 0$ case is similar. With $\alpha = 1$ we are willing to accept making errors with probability 1 when the null hypothesis is true. We therefore should be able to maximize power, and obtain a UMP test, by always rejecting regardless of the value of X. This is acheived by setting $c = 0$ and $\gamma = 1$ for part i) of the proof. This amounts to

$$T_*(X) = \begin{cases} 1 & f_1(X) \geq 0 \\ 0 & f_1(X) < 0 \end{cases} = 1$$

The second equality is justified by the fact that densities must be non-negative. To show this is the UMP level $\alpha = 1$ test, we have to show that this test is a) level $\alpha = 1$ and b) has at least as much power as every level $\alpha = 1$ test. For a) we have

$$\beta_{T_*}(P_0) = P_{P_0}(T_*(X) = 1) = 1 = \alpha$$

For b) we have

$$\beta_{T_*}(P_1) = P_{P_1}(T_*(X) = 1) = 1 \geq P_{P_1}(T(X) = 1) = 1 = \beta_T(P_1)$$

for any test T.

Now need to show ii). Suppose T_{**} is UMP size $\alpha = 1$. Then $1 = \beta_{T_{**}}(P_0) = P_{P_0}(T_{**} = 1) = 1$. Thus

$$T_{**}(X) = \begin{cases} 1 & f_1(X) \geq 0 \\ 0 & f_1(X) < 0 \end{cases} a.s. P_0$$

We saw earlier that UMP size 1 T_* had power 1 under P_1. Since T_{**} is UMP it must also have power 1 under P_1. Therefore $1 = \beta_{P_1}(T_{**}) = P_{P_1}(T_{**} = 1) = 1$. Thus

$$T_{**}(X) = \begin{cases} 1 & f_1(X) \geq 0 \\ 0 & f_1(X) < 0 \end{cases} a.s. P_1$$

For $\alpha = 0$ case, set $c = \infty$ and $\gamma = 0$.

Section 6.6 Exercise 3:

a) Let T^{α_1} and T^{α_2} be the size α_1 and α_2 UMP tests defined by Theorem 6.1 i) Equation 6.3. These are defined by c and γ which we write as functions of α. Thus T^{α_1} is defined by $c(\alpha_1)$ and $\gamma(\alpha_1)$ while T^{α_2} is defined by $c(\alpha_2)$ and $\gamma(\alpha_2)$. Goal is to show that $c(\alpha_2) \leq c(\alpha_1)$. This result intuitively makes sense because by assumption $\alpha_1 < \alpha_2$ so T^{α_2} has more type I error and thus rejects more often. By setting c smaller, a test will reject for a larger set of X.

To formally show this, note that $\beta_{T^{\alpha_2}}(P_0) > \beta_{T^{\alpha_1}}(P_0)$. Thus

$$
\begin{aligned}
&P_{P_0}(f_1(X) > c(\alpha_2)f_0(X)) + \gamma(\alpha_2)P_{P_0}(f_1(X) = c(\alpha_2)f_0(X)) \\
>&P_{P_0}(f_1(X) > c(\alpha_1)f_0(X)) + \gamma(\alpha_1)P_{P_0}(f_1(X) = c(\alpha_1)f_0(X))
\end{aligned} \tag{1}
$$

Proceed with proof by contradiction. Suppose $c(\alpha_2) > c(\alpha_1)$. Then

$$
\begin{aligned}
&P_{P_0}(f_1(X) > c(\alpha_2)f_0(X)) + \gamma(\alpha_2)P_{P_0}(f_1(X) = c(\alpha_2)f_0(X)) \\
\leq&P_{P_0}(f_1(X) \geq c(\alpha_2)f_0(X)) \\
\leq&P_{P_0}(f_1(X) > c(\alpha_1)f_0(X)) \\
\leq&P_{P_0}(f_1(X) > c(\alpha_1)f_0(X)) + \gamma(\alpha_1)P_{P_0}(f_1(X) = c(\alpha_1)f_0(X))
\end{aligned}
$$

But this contradicts Equation 1.

b) Showing Type II error probability of T^{α_1} is larger than T^{α_2} is equivalent to showing its power is smaller under the alternative, i.e. $\beta_{T^{\alpha_2}}(P_1) > \beta_{T^{\alpha_1}}(P_1)$. Follow the reasoning in proof of Theorem 6.1 i). Note that

$$[T_{\alpha_2}(x) - T_{\alpha_1}(x)][f_1(x) - c(\alpha_2)f_0(x)] \geq 0$$

following along the same reasoning of the theorem. Then integrating wrt measure ν we have

$$\beta_{T^{\alpha_2}}(P_1) - \beta_{T^{\alpha_1}}(P_1) \geq c(\alpha_2)(\alpha_2 - \alpha_1)$$

By assumption $\alpha_2 - \alpha_1 > 0$. Since $\alpha_2 > \alpha_1 \geq 0$ we have that $c(\alpha_2) > 0$. Thus the rhs is positive and $\beta_{T^{\alpha_2}}(P_1) > \beta_{T^{\alpha_1}}(P_1)$.

Section 6.6 Exercise 5:

a

It is useful to draw out the densities f_{θ_1} and f_{θ_2} and consider which values of x most support H_0 or H_a.

Since these are densities wrt Lebesgue measure, the UMP test can be non–randomized. First we find the general form of T by finding x which $T(x) = 1$. By the Neyman-Pearson Lemma (Theorem 6.1), this is $f_{\theta_1}(x) > cf_{\theta_0}(x)$.

$$2\theta_1^{-2}(\theta_1 - x)\mathbf{1}_{(0,\theta_1)}(x) > c2\theta_0^{-2}(\theta_0 - x)\mathbf{1}_{(0,\theta_0)}(x)$$

Solving for x we obtain

$$x > \frac{c/\theta_0 - 1/\theta_1}{c/\theta_0^2 - 1/\theta_0^2}$$

Thus one rejects for large x. For a size α test, find what the appropriate threshold t is:

$$
\begin{aligned}
\alpha &= P_{\theta_0}(X > t) \\
&= \int_t^{\theta_1} 2(\theta_1^{-2}(\theta_1 - x))dx \\
&= 1 - 2t/\theta_1 + t^2/\theta_1^2
\end{aligned}
$$

By the quadratic formula we get $t = \theta_0(1 - \sqrt{\alpha})$. Thus a UMP size α test is

$$T(x) = \begin{cases} 1 & x > \theta_0(1 - \sqrt{\alpha}) \\ 0 & o.w. \end{cases}$$

Section 6.6 Exercise 6:

a

Note that

$$\frac{f_1(x)}{f_0(x)} = \begin{cases} 0 & \theta_0 < x_{(1)} \leq \theta_1 \\ e^{n(\theta_1 - \theta_0)} & x_{(1)} > \theta_1 \end{cases}$$

where $x_{(1)} = \min x_i$. The likelihood ratio can only take 2 unique values, thus we will need a randomized test function (more formally, if $B = \{x : f_1(x) = cf_0(x)\}$, $\nu(B) \neq 0$ for some c). This is unusual for continuous random variables. The possible forms for the UMP are

$$T(x) = \begin{cases} 0 & \theta_0 < x_{(1)} \leq \theta_1 \\ \gamma & x_{(1)} > \theta_1 \end{cases}$$

and

$$T'(x) = \begin{cases} \gamma & \theta_0 < x_{(1)} \leq \theta_1 \\ 1 & x_{(1)} > \theta_1 \end{cases}$$

Note that the pdf of $X_{(1)}$ is $f(x_{(1)}) = ne^{-n(x_{(1)} - \theta)} \mathbb{1}_{(\theta, \infty)}(x_{(1)})$. For small α we do not reject when $f_1/f_0 = 0$ and probabilistically reject for $f_1/f_0 > 0$ (i.e. test is of form T). Consider

$$\begin{aligned} P_0(f_1(X)/f_0(X) = 0) &= P_0(f_1(X) = 0) \\ &= P_0(X_{(1)} \in (\theta_0, \theta_1)) \\ &= \int_{\theta_0}^{\theta_1} ne^{-n(x - \theta_0)} dx \\ &= 1 - e^{n(\theta_0 - \theta_1)} \end{aligned}$$

Thus $P_0(f_1(X)/f_0(X) = e^{n(\theta_1 - \theta_0)}) = e^{n(\theta_0 - \theta_1)}$.

<u>Case 1: $\alpha \leq e^{n(\theta_0 - \theta_1)}$</u>, Set $\gamma = \alpha / e^{n(\theta_0 - \theta_1)}$ and use form T.

<u>Case 2: $\alpha > e^{n(\theta_0 - \theta_1)}$</u>, Set

$$\gamma = \frac{\alpha - e^{n(\theta_0 - \theta_1)}}{1 - e^{n(\theta_0 - \theta_1)}}$$

and use form T'.

Section 6.6 Exercise 14:

a

This is an exponential model in scale parameterization. So it is part of the exponential family and we can use Example 6.3 to find an MLR and Theorem 6.2 to construct a UMP test.

$$f_\theta(x) = \exp\{(-1/\theta)x + \log\theta^{-1}\}\mathbb{1}_{(0,\infty)}(x)$$

So

$$f_\theta(\vec{x}) = \exp\{(-1/\theta)\sum x_i + n\log\theta^{-1}\}\mathbb{1}_{(0,\infty)}(\vec{x})$$

By Example 6.3 $Y = \sum X_i$ is an MLR. Note that

- $Y \sim Gamma(n, \theta)$ in shape-scale parameterization (sum of exponentials is gamma)
- Y is continuous, so can use non–randomized test

By Theorem 6.2 we have

$$T(x) = \begin{cases} 1 & Y(x) \geq c \\ 0 & Y(x) < c \end{cases}$$

To control Type I error at α we have $\alpha = P_{\theta_0}(Y \geq c)$. Thus

$$c = F^{-1}(1-\alpha)$$

where F^{-1} is the $Gamma(n, \theta_0)$ quantile function in shape-scale parameterization.

b

The density should read $f_\theta(x) = \theta x^{\theta-1}\mathbb{1}_{(0,1)}(x)$. The solution is similar to part a.

$$f_\theta(x) = \exp\{(\theta-1)\log x + \log\theta\}\mathbb{1}_{(0,\infty)}(x)$$

So

$$f_\theta(\vec{x}) = \exp\{(\theta-1)\sum\log(x_i) + n\log\theta\}\mathbb{1}_{(0,\infty)}(\vec{x})$$

By Example 6.3 $Y = \sum\log(x_i)$ is an MLR. Note that $\log x_i$ is $Expo(1/\theta)$ distributed (scale parameterization). Thus Y is $Gamma(n, 1/\theta)$ (shape-scale parameterization). Since Y is continuous, we will use a non–randomized test. Thus by Theorem 6.2 we have

$$T(x) = \begin{cases} 1 & Y(x) \geq c \\ 0 & Y(x) < c \end{cases}$$

To control Type I error at α we have $\alpha = P_{\theta_0}(Y \geq c)$. Thus

$$c = F^{-1}(1-\alpha)$$

where F^{-1} is the $Gamma(n, 1/\theta_0)$ quantile function in shape-scale parameterization.

Section 6.6 Exercise 15:

For a given α the UMP test by Theorem 6.2 i) is

$$T_\alpha(x) = \begin{cases} 1 & Y(x) > c(\alpha) \\ 0 & Y(x) \leq c(\alpha) o.w. \end{cases}$$

We do not need the equality case because $Y(X)$ is continuous r.v. Note that $c(\alpha)$ is a continuous, decreasing function and thus invertible.

Next we have the p–value is

$$\begin{aligned}
\widehat{\alpha} &= \inf_{\alpha}\{\alpha \in (0,1) : T_\alpha(y) = 1\} \\
&= \inf_{\alpha}\{\alpha \in (0,1) : y > c(\alpha)\} \\
&= c^{-1}(y)
\end{aligned}$$

where the last equality is justified because c is a decreasing function of α and continuous / invertible. Then we have

$$\begin{aligned}
\widehat{\alpha} &= \sup_{\theta \leq \theta_0} \mathbb{E}_\theta[T_{\widehat{\alpha}}(X)] \\
&= \mathbb{E}_{\theta_0}[T_{\widehat{\alpha}}(X)] \\
&= P_{\theta_0}(Y(X) > c(\widehat{\alpha})) \\
&= P_{\theta_0}(Y(X) > c(c^{-1}(y))) \\
&= P_{\theta_0}(Y(X) > y)
\end{aligned}$$

The second equality is justified by Lemma 6.3.