Rice STAT 533/ GS01 1283 Homework 3 Solutions
February 8, 2020
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Section 6.6 Exercise 101:

a) The parameters in the multinomial model are $p = (p_{11}, p_{12}, p_{21})$ where $p_{22} = 1 - p_{11} - p_{12} - p_{21}$. The density wrt counting measure is

$$f_p(X) = \frac{n!}{x_{11}!x_{12}!x_{21}!x_{22}!} p_{11}^{x_{11}} p_{12}^{x_{12}} p_{21}^{x_{21}} (1 - p_{11} - p_{12} - p_{21})^{x_{22}}$$

The null hypothesis is $P(A) = P(B)$ which is equivalent to $p_{11} + p_{21} = p_{11} + p_{12}$ which is equivalent to $p_{12} = p_{21}$. The full model has $k = 3$ free parameters. Following equation 6.63 in the textbook we can specify the null hypothesis as $H_0 : p = g(\phi)$ where $\phi = (\phi_1, \phi_2)^T$ and $g(\phi) = (\phi_1, \phi_2, \phi_2)^T$. Thus the domain of g has $k - r = 2$ dimensions and $r - 1$. So $-2 \log \lambda_n \sim \chi_1^2$ by Theorem 6.5.

To find the form of the LR statistic, note that the unrestricted MLE is $\widehat{p}_{11} = X_{11}/n$, $p_{12} = X_{12}/n$ and $p_{21} = X_{21}/n$. The restricted MLE is $\widetilde{p}_{11} = X_{11}/n$ and $\widetilde{p}_{12} = \widetilde{p}_{21} = (X_{12} + X_{21})/(2n)$. Thus the LRT is

$$\lambda_n = \frac{(\frac{X_{12}+X_{21}}{2})^{X_{12}+X_{21}}}{X_{12}^{X_{12}} X_{21}^{X_{21}}}.$$

b) Following the multinomial parameterization of a), we can let $R(p) = (0, 1, -1)p = p_{12} - p_{21}$ (following Equation 6.64 specification of H_0 in textbook). Then $C = \frac{\partial R}{\partial p} = (0, 1, -1)^T$. Following Exercise 100, we can determine

$$I_n(p) = n \begin{pmatrix} p_{11}^{-1} & 0 & 0 \\ 0 & p_{12}^{-1} & 0 \\ 0 & 0 & p_{21}^{-1} \end{pmatrix} + (n/p_{22})\mathbf{1}$$

where $\mathbf{1}$ is a 3×3 matrix of 1s. By Sherman Woodbury Morrison we have

$$I_n(p) = \frac{1}{n} \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & p_{12} & 0 \\ 0 & 0 & p_{21} \end{pmatrix} + \frac{1}{n} \begin{pmatrix} p_{11}^2 & p_{11}p_{12} & p_{11}p_{21} \\ p_{11}p_{12} & p_{12}^2 & p_{12}p_{21} \\ p_{11}p_{21} & p_{21}p_{12} & p_{21}^2 \end{pmatrix}$$

After some matrix multiplication we obtain

$$W_n(\widehat{p}) = \frac{n(X_{12} - X_{21})^2}{n(X_{12} - X_{21}) - (X_{12} - X_{21})^2}$$

For the Rao test, note that the score function is

$$s(p) = \frac{\partial l}{\partial p} = (X_{11}/p_{11}, X_{12}/p_{12}, X_{21}/p_{21}) - (X_{22}/(1 - p_{11} - p_{12} - p_{21}))(1, 1, 1)^T$$

Plugging $\widetilde{p}$ into s we obtain

$$R_n = s(\widetilde{p})^T I_n^{-1}(\widetilde{p}) s(\widetilde{p}) = \frac{(X_{12} - X_{21})^2}{X_{12} + X_{21}}$$

Section 6.6 Exercise 105:

a) Let $X = \sum X_i$. Define the Beta density at θ with parameters a and b as $B(\theta, a, b)$. By conjugacy the posterior distribution on θ is the beta density $B(\cdot, a+X, b+n-X)$. Thus

$$\widehat{\pi}_0 = \int_0^{\theta_0} B(\theta, a+X, b+n-X)d\theta$$

and $\widehat{\pi}_1 = 1 - \widehat{\pi}_0$. Reject the null if $\widehat{\pi}_0 < 1/2$. The prior probability of the null is

$$\pi_0 = \int_0^{\theta_0} B(\theta, a, b)d\theta$$

and $\pi_1 = 1 - \pi_0$. Thus the Bayes factor is

$$\beta = \frac{\frac{\widehat{\theta}_0}{\widehat{\theta}_1}}{\frac{\theta_0}{\theta_1}}$$

b) For the Bayes factor we can compute $m_1(x) = P(x|H_1)$ and $m_0(x) = P(x|H_0)$. This is

$$\begin{aligned} m_1(x) &= \int_{\theta=0}^{1} \binom{n}{x} \theta^x (1-\theta)^{n-x} B(\theta, a, b)d\theta \\ &= \binom{n}{x} \frac{B(x+a, n-x+b)}{B(a,b)} \end{aligned}$$

where the second inequality is justified by the fact that this is a beta–binomial distribution (distribution of bernoulli random variable where probability of success is drawn from a beta distribution).

Since $m_0(x) = \binom{n}{x}\theta_0^x(1-\theta_0)^{n-x}$. Thus the Bayes factor is

$$\beta = \frac{m_0(x)}{m_1(x)}$$

Note that

$$\frac{\widehat{\pi}_0}{1-\widehat{\pi}_0} = \beta\frac{\pi_0}{1-\pi_0}$$

Thus

$$\widehat{\pi}_0 = \frac{(\pi_0/(1-\pi_0))\beta}{1+(\pi_0/(1-\pi_0))\beta}$$