Distances in Cosmology

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1 An expanding universe

In an expanding universe, distances between two points or events increases with time. We start by considering a spherical sphere which may be radially expanding or contracting. A galaxy located at coordinate $\mathbf{x}$ at time t_0 will be located at a different position $\mathbf{r}(\mathrm{t})$ at time t, and their relations are given by the following relation

$$\mathbf{r}(t) \; = \; a(t, \mathbf{x})\mathbf{x}. \tag{1}$$

If we assume that the universe is homogeneous, *i.e.*, is identical in all directions and positions, the above equation reduces to

$$\mathbf{r}(t) \; = \; a(t)\mathbf{x}. \tag{2}$$

The function $a(t)$ depends only on time, and is called the cosmic scale factor. It obeys $a(t_0) \; = \; 1$. t_0 is in principle arbitrary. We choose it to be today.

Galaxies that move according to equation (2) are called *comoving* galaxies, and $\mathbf{x}$ is the comoving coordinates. The world line ($\mathbf{r}$, t) of a comoving observer is uniquely defined by $\mathbf{x}$, $(\mathbf{r}, t) \; = \; (a(t)\mathbf{x}, t)$.

2 Hubble Constant

The velocity of such a comoving galaxy is

$$\mathbf{v}(\mathbf{r}, t) \; = \; \frac{d}{dt}\mathbf{r}(t) \; = \; \frac{da}{dt}\mathbf{x} \; \equiv \; \dot{a}\mathbf{x} \; = \; \frac{\dot{a}}{a}\mathbf{r} \; \equiv \; H(t)\mathbf{r}. \tag{3}$$

At $t \; = \; t_0$, $H_0 \; = \; H(t_0)$, which describes the law of expansion of the universe at epoch t_0.

Measured values of H_0 is around 70 km/sec/Mpc.

3 Newtonian cosmology

The scaling parameter a(t) is can be determined from the basic laws of physics. The conservation of energy requires that the sum of kinetic and potential energy to be constant in time

$$\frac{mv^2(t)}{2} - \frac{GMm}{r(t)} = -Kc^2 f(\mathbf{x}), \tag{4}$$

where $f(\mathbf{x})$ is an arbitrary function of $\mathbf{x}$, $v(t) = \dot{a}(t)\mathbf{x}$, and $r(t) = a(t)x$. It can be seen that $f(x) = x^2/2$ so that a and its derivatives are independent of $\mathbf{x}$. This equation leads to

$$\dot{a}^2 = \frac{8\pi G}{3}\rho_0 \frac{1}{a} - Kc^2 = \frac{8\pi G}{3}\rho(t)a^2(t) - Kc^2; \tag{5}$$

here, Kc^2 is a constant of integration that is also a measure of the total energy of the comoving galaxy. Thus, the history of expansion of the galaxy depends on K. The sign of K characterizes the qualitative behavior pf the cosmic expansion history.

- If $K < 0$, the right hand of eq (5) is always positive. Since $\dot{a}/a$ is positive today, it would be positive for all t, the universe expands forever. The universe is said to be open
- If $K = 0$, the right hand of eq (5) is always positive, the universe will expand forever, but $\dot{a} \to 0$ for $t \to \infty$ The universe is said to be flat.
- If $K > 0$, the right hand side of eq (5) vanishes when $a = a_{max} = (8\pi G\rho_0)/(3Kc^2)$. For this value of a, $da/dt = 0$, and the expansion will come into an halt. After that the universe will contract and re-collapse. The universe is said to be closed.

3.1 The critical density

In the special case of $K = 0$, the density of the universe at the current time t_0 is called the critical density

$$\rho_{cr} \equiv \frac{3H_0^2}{8\pi G} = 1.88 \times 10^{-29} h^2 g/cm^3 \tag{6}$$

It is convenient to define a dimensionless density parameter

$$\Omega_0 \equiv \frac{\rho_0}{\rho_{cr}}; \tag{7}$$

where $K > 0$ corresponds to $\Omega_0 > 1$, $K < 0$ corresponds to $\Omega_0 < 1$, and $K = 0$ corresponds to $\Omega_0 = 1$.

4 General relativistic cosmology

Major modifications to Newtonian cosmology

- It is not mass. Mass and energy are equivalent, so the equation should include all forms of energy.
- The interpretation of the expansion is different. The galaxies and observers are NOT moving away from each other, nor is the universe an expanding sphere. Instead, it is space itself that expands. In particular, the redshift is not Doppler redshift, but is a property of expanding spacetimes.

4.1 The Friedmann-Lamaitre-Robertson-Walker Expansion Equations

$$\begin{aligned}\left(\frac{\dot{a}}{a}\right) &= \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda}{3} \\ \frac{\ddot{a}}{a} &= \frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda}{3}\end{aligned}$$

Λ is Einstein's cosmological constant.

4.2 The solution

The equation above can be solved to yield

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = H_0^2[a^{-4}(t)\Omega_r + a^{-3}(t)\Omega_m + a^{-2}(t)(1 - \Omega_m - \Omega_\Lambda) + \Omega_\Lambda]$$

4.3 Redshift

$$1 + z = \frac{a_0}{a} = \frac{1}{a}$$

4.4 Local Hubble Law

$$z = \frac{H_0}{c}D$$

5 Distances in cosmology

How can a measurement of redshift be translated into a distance? What distance?

5.1 Angular-diameter distance

$$D_A(z) \equiv \sqrt{\pi R^2/\omega};$$

where R is the radius of the galaxy, and ω is the measured solid angle of the galaxy.

5.2 Luminosity distance

$$D_L(z) = \sqrt{L/(4\pi S)};$$

where L is the luminosity of the galaxy, and S is the flux measured of the galaxy.

The two distances agree for $z << 1$. A general relation exists

$$D_L(z) = (1+z)^2 D_A(z)$$

5.3 A general solution

The general solution to the Friedmann-Lamaitre equations is

$$\mu(z, \Omega_M, \Omega_\Lambda, H_0) = 5log_{10}(d_L(z, \Omega_M, \Omega_\Lambda, H_0)) + 25,$$

where the luminosity distance d_L is given by

$$d_L(z, \Omega_M, \Omega_L, H_0) = \frac{c(1+z)}{H_0\sqrt{|k|}} \times \mathbf{E}\left(\sqrt{|k|}\int_0^z [(1+z')^2(1+\Omega_M z') - z'(2+z')\Omega_\Lambda]^{-1/2} dz'\right),$$

where $k = 1 - \Omega_M - \Omega_\Lambda$, and $\mathbf{E}$ is defined as

$$\mathbf{E}(x) = \begin{cases} \sin(x) & \text{for } k<0 \\ x & \text{for } k = 0 \\ \sinh(x) & \text{for } k>0 \end{cases}$$